Introductio

in Elementarem Functionum Ellipticarum Theoriam,

quam

Cons. Ampliss. Philos. Ord. Lundens.,

Præside

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Pro Laurea

p. p.

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Quando igitur a & b parum differunt, erit quam proxime $\varphi = \frac{\gamma}{\sqrt{ab}}$. Universim vero videmus, postremam formulam cum hac (p. 55 occurente)

$$d\varphi' = \frac{d\alpha'}{\sqrt{b_{,}^2 + c_{,}^2 S^0 \alpha'^2}}$$

comparando & quidem in specie $\alpha' = 2\gamma$, $b_{,} = 2\sqrt{ab}$, $c_{,} = a - b$, ideoque $a' = \sqrt{c_{,}^2 + b_{,}^2} = a + b$, ponendo, illum sectorem φ alteri φ', cujus radii sunt $a + b$ & $2\sqrt{ab}$ atque angulus minor (α') duplus medii prioris (γ), æquari. Radii vero ita ad æqualitatem appropinquant in eadem ratione, ac medium arithmeticum atque geometricum; ideoque hanc transformationem sæpius adhibendo sectorem hypercyclicum tandem per circularem exprimere licebit, id quod satis cito evenit. Ipsam igitur aliquantum explanamus. Posuimus $T = a\,T^0\alpha = b\,.\,T^0\beta = R\,.\,T^0\gamma$, & $T = T\left(\frac{a}{b}, \varphi\right)$, similiterque ponamus $T' = a'\,T^0\alpha' = b'\,T^0\beta' = R'.\,T^0\gamma' = T\overbrace{\frac{a'}{b'}, \varphi'}$, eritque

$$d\varphi' = \frac{dT'}{\sqrt{(a'^2 + T'^2)(b'^2 + T'^2)}} = \frac{d\alpha'}{\sqrt{b'^2 + c'^2 S\alpha'^2}} = \frac{d\beta'}{\sqrt{a'^2 - c'^2 . S^0\beta'^2}}$$

atque $d\varphi' = \frac{d\,.\,\overset{\frown}{2\gamma'}}{\sqrt{4\,.\,a'b' + \overline{a'-b'}^2\,\overline{S^0 2\gamma'}^2}}$; & simul $\alpha' = 2\gamma$, $b' = 2\sqrt{ab}$, $c' = a - b$, & $a' = a + b$ sit; eritque $T^0\alpha' = T^0\overset{\frown}{2\gamma}$, $T^0\beta' = \frac{a'}{b'}\,.\,T^0\alpha' = \frac{a+b}{2\sqrt{ab}}\,.\,T^0(2\gamma)$, existente $T^0\gamma = \frac{b}{R}\,.\,T^0\beta = \sqrt{\frac{b}{a}}\,.\,T^0\beta$. Ut igitur a formula $\varphi = \int_0 \frac{d\beta}{\sqrt{a^2 - c^2 S^0 \beta^2}}$ ad similem $\varphi' = \int_0 \frac{d\beta'}{\sqrt{a'^2 - c'^2 S^0 \beta'^2}}$ progrediatur, calculandi sunt anguli γ & β' ex formulis

$$1)\ T^0\gamma=\sqrt{\frac{b}{a}}\,.\,T^0\beta \text{ seu } 1)\ Tang\,\gamma=\sqrt{\frac{b}{a}}\,.\,Tang\,\beta$$

$$2)\ T^0\beta'=\frac{a+b}{2\sqrt{ab}}\,.\,T^0\,2\gamma'..\ 2)\ Tang\beta'=\frac{a+b}{2\sqrt{ab}}\,.\,Tang\,2\gamma,$$ et præterea radii $a'=a+b$ atque $c'=a-b$. Patet vero has formulas simplicissimas inverso ordine adhiberi posse. Præbent nempe $a=\frac{1}{2}(a'+c')$, $b=\frac{1}{2}(a'-c')$, ideoque $2\sqrt{ab}=\sqrt{a'^2-c'^2}=b'$, $T^0(2\gamma)=\frac{b'}{a'}\,.\,T^0\beta'$ atque $T^0\beta=\frac{a'+c'}{b'}\,.\,T^0\gamma$. Præterea si $c=a.e$ & $c'=a'.e'$ ponimus, erit $\frac{b}{a}=\sqrt{1-e^2}$, & $e'=\frac{c'}{a'}=\frac{a-b}{a+b}=\frac{1-\sqrt{1-e^2}}{1+\sqrt{1-e^2}}=\frac{\overparen{1-\sqrt{1-e}}^2}{2e^2}$, atque inverse $\sqrt{1-e^2}=\frac{b}{a}=\frac{1-e'}{1+e'}$, ideoque $e^2=\frac{\overparen{1+e'}^2-\overparen{1-e'}^2}{\overparen{1+e'}^2}=$ $=\frac{4e'}{\overparen{1+e'}^2}$ seu $e=\frac{2\sqrt{e'}}{1+e'}$. His igitur valoribus positis, erit $\frac{1}{a}F\overparen{e,\beta}=$ $=\varphi=\varphi'=\frac{1}{a'}\cdot F\overparen{e',\beta'}$, seu $F\overparen{e,\beta}=\frac{1+e'}{2}\cdot F\overparen{e',\beta'}$, existente $T^0\,2\gamma=$ $=\sqrt{1-e'^2}\,.\,T^0\beta'$ atque $T^0\beta=\frac{1+e'}{\sqrt{1-e'^2}}.T^0\gamma$ seu $T^0\beta=\sqrt{\frac{1+e'}{1-e'}}\,.\,T^0\gamma$.

Et hæc est transformatio, quæ, ut Cel. Jacobi loquitur, ad dyadem pertinet, & jam a Cel. LeGendre pertractata, qui tamen æquationem auxiliarem habet: $S^0(2\beta-\beta')=e'.S^0\beta'$. Plurimas vero, easque altioris indaginis hi celeberrimi viri invenerunt. Illa vero in nostris signis ita breviter exprimitur:

$$T\left(\frac{2\sqrt{ab}}{a+b},\ \varphi\right)=\frac{2(a+b)\sqrt{ab}\,.\,T\left(\frac{a}{b},\varphi\right)}{ab-\left(T\overbrace{\frac{a}{b},\varphi}\right)^2}\ \text{seu}\ T\overbrace{\frac{b'}{a'},\varphi}=\frac{a'b'T\overbrace{\frac{b}{a},\varphi}}{ab-\overbrace{T\frac{b}{a},\varphi}^{\,2}}.$$

Posuimus enim $T\overbrace{\frac{a}{b},\varphi}=\sqrt{ab}\,T^0\gamma$, atque invenimus $a'=a+b$, $b'=2\sqrt{ab}$, $T\left(\frac{a'}{b'},\ \varphi'\right)=a'.T^0(2\gamma)$, & $\varphi=\varphi'$. Erit igitur

$$T\left(\frac{2\sqrt{ab}}{a+b},\varphi\right)=T\left(\frac{b'}{a'},\ \varphi\right)=T\left(\frac{a'}{b'},\ \varphi\right)=a'\,T^0\,(2\gamma),\ \text{ideoque}$$

$$T\left(\frac{2\sqrt{ab}}{a+b},\varphi\right)=(a+b).\frac{2\,T^0\gamma}{1-\overbrace{T^0\gamma}^{\,2}},\ \text{ubi valor}\ \frac{T\left(\frac{a}{b},\varphi\right)}{\sqrt{ab}}\ \text{ipsius}\ T^0\gamma$$

tantum substituendus est, quo formula proposita exsultet. Inde & formulas ad reliquas functiones transformandas invenies, attamen multo complicatiores, siquidem ad eosdem modulos referendæ sint; sin vero ad alios, substitutio imaginaria multum adjuvat. Sic ex. gr. ponendo $\varphi=\iota\psi$ *) erit ex formula $T\overbrace{\frac{b}{a},\varphi}=\iota\,S\left(\frac{a}{c},\varphi\iota\right)$, præcedenti conjuncta, $S\left(\frac{a'}{c'},\psi\right)=$

$$=S\left(\frac{a'}{c'},\varphi\iota\right)=\frac{1}{\iota}\,T\left(\frac{a'}{b'},\ \varphi\right)=\frac{1}{\iota}.\frac{a'b'.\iota\,S\overbrace{\frac{a}{c},\ \psi}}{ab+\overbrace{S\frac{a}{c},\ \psi}^{\,2}},\text{seu}\ S\left(\frac{a+b}{a-b},\psi\right)=S\left(\frac{a'}{c'},\psi\right)=$$

$$=\frac{a'b'.\,S\overbrace{\frac{a}{c},\ \psi}}{ab+\overbrace{S\frac{a}{c},\ \psi}^{\,2}};\ \&\ \text{similiter}\ S\left(\frac{a+c}{a-c},\ \psi\right)=\frac{2\,\overbrace{a+c}\sqrt{ac}\,.\,S\overbrace{\frac{a}{b},\ \psi}}{ac+\overbrace{S\frac{a}{b},\ \psi}^{\,2}}.$$

Et ad sinum minorem formulam isti haud dissimilem invenies. Hæc functionum nostrarum secundum radios transformatio haud spernendi estmo-

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*) Si ι imaginarium $\sqrt{-1}$ significat.

menti, quia illius ope horum ratio $\left(\frac{a}{c}\right)$ inter limites satis arctos includitur. Ponamus enim brevitatis caussa $a = 1 = a'$ (debita nempe secundum p. 40 facta substitutione), & permutemus c & c' ut $e = c'$ & $e' = c$ evadat, eritque $c' = \frac{2\sqrt{c}}{1+c}$. Jam vero si fuerit $c = o$, erit $c' = o$, sin vero $c = 0{,}07$ fuerit, erit $c' = 0{,}4945$, & igitur si $o < c < 0{,}07$ fuerit, erit etiam....$0{,}0.. < c' < 0{,}4945$. Si igitur functionum nostrarum valores ad eccentricitatem $c < 0{,}07$ (Ex. gr. tabulæ cujusdam ope) habentur, istarum formularum ope usque ad $c = 0{,}4945$ calculantur. Deinde si fuerit $c = > 0{,}4945$, erit $c' = > 0{,}9411$, atque $c' = 1$, quando $c = 1$; loco ipsius c igitur inter $0{,}4945$ & $0{,}9411$ adhiberi potest c' inter $0{,}9411$ & 1. Istius igitur transformationis ope functiones hypercyclicæ rediguntur ad similes, quarum eccentricitates vel inter limites 0 & $0{,}07$ vel inter $0{,}9411$ & 1 versantur, vel etiam accuratius inter limites $0....0{,}06957$ atqne $0{,}9406......1$ datæ sunt; ita enim tabulæ necessariæ ambitus quam minimus evadit. His prope accedunt limites $0...$ & $7 - 4\sqrt{3}$ atque $\frac{2}{3}\sqrt{2}$ & 1, quorum priores ad $c' < \frac{1}{2}$, & posteriores ad $c > \frac{1}{2}$ in usum vocandi sunt. Posito vero $c = S^\circ(\vartheta)$, limites isti erunt $0^\circ < \vartheta < 9^\circ 53'$, cum $0^\circ < \vartheta' < 45^\circ$, atque $80^\circ 7' < \vartheta' < 90^\circ$, cum $45^\circ < \vartheta < 80^\circ$ l. 90°. (Est vero hujus anguli $9^\circ 53'$ sinus trig. $= (\sqrt{2} - 1)^2$). Ita igitur angulus (ϑ) eccentricitatis vel $< 10^\circ$, vel $> 80^\circ$ efficitur. Si igitur tabula, qualis Ex. gr. p. 58 depingitur, ad hos modulos constructa fuerit, per hanc transformationem valores functionum Ell. cujuscunque moduli computabuntur. Hi vero limites moduli multo ampliores sunt ac ii, quos in prima transformatione invenimus. At præsens transformatio iterata eodem redit. Si enim ponimus $c = \varkappa^2$, & similiter $c' = \varkappa'^2$, erit $\varkappa'^2 = \frac{2\varkappa}{1+\varkappa^2}$, & ite-

rum ponendo $\varkappa''^2 = \frac{2\varkappa'}{1+\varkappa'^2}$, erit $\varkappa''^2 = \frac{2\sqrt{\frac{2\varkappa}{1+\varkappa^2}}}{(1+\varkappa)^2:(1+\varkappa^2)}$ seu $(\varkappa'')^4 = \frac{8\varkappa(1+\varkappa^2)}{(1+\varkappa)^4}$. Quoniam vero hic valor omnino convenit cum valore ipsius k^4 antea invento, hæc transformatio dyadica bis adhibita ad eandem ducit, quam antea per duas substitutiones imaginarias invenimus, quarum una præbebat modulorum rationem $\varkappa' = \frac{1-\varkappa}{1+\varkappa}$, alteraque $\varkappa'^4 + k^4 = 1$. Sin vero in prima transformatione facimus $x = \varkappa S\varphi$ atque $x' = -kS\psi$, ideoque $\varkappa S\varphi = \frac{\varkappa' - \sqrt{1-k^4 S\psi^2}}{\varkappa' + \sqrt{1-k^4 S\psi^2}}$, erit illa $\frac{(1-\varkappa^2)\,d\varphi}{\sqrt{1-\varkappa^4 S\varphi^2}} = \frac{-2.\varkappa'.d\psi}{\sqrt{1-k^4 S\psi^2}}$, ideoque $F(\varkappa^2,\varphi) = \frac{-2}{(1+\varkappa)^2}.F(k^2,\psi) + F\left(\varkappa^2,\frac{\pi}{2}\right)$, (constante sc. ex casu $\psi = o$, $\varphi = \frac{1}{2}\pi$ determinata). Jam vero p. 56 & 39 vidimus, $F\left(\varkappa^2,\frac{\pi}{2}\right) - F(\varkappa^2,\varphi)$ in $F(\varkappa^2,\alpha)$ colligi, si $T\alpha.T\varphi = \frac{a}{b} = \frac{1}{\sqrt{1-\varkappa^4}}$ fuerit; ideoque erit $F(\varkappa^2,\alpha) = \frac{2}{(1+\varkappa)^2}\cdot F(k^2,\psi)$. Præsens vero transformatio bis adhibita præbet $F(e,\beta) = \frac{1+e'}{2}.F(e',\beta') = \frac{1+e'}{2}.\frac{1+e''}{2}.F(e'',\beta'')$, exsistente $e = \frac{2\sqrt{e'}}{1+e'}$ & $e' = \frac{2\sqrt{e''}}{1+e''}$, ideoque posito $e'' = \varkappa^2$, & $e = k^2$, functionum transcendentium ratio erit $\overline{(1+e'}.\overline{1+e''}:4 =) \overline{1+\varkappa}^2:4$, seu dimidia ipsarum in prima. Hoc vero ad formulas duplicandi neccessario ducit, sc. $F(e,\psi) =$

$= 2F(e,\beta)$, (si $\alpha = \beta''$ accipitur) vel $F(e'', \beta'') = 2F(e'', \alpha)$, (si $\psi = \beta$), ad quas igitur modulum bis mutando, quo idem resurgit, pervenitur. Dantur & plures modulum variandi modi, in præsenti vero hos omittimus, in sectorum rationem accuratius inquisituri.

Sect. III. *Comparatio functionum Ellipticarum secundum sectores et amplitudines.*

Vidimus jam functiones nostras secundum modulos & realiter & imaginarie variari posse; at secundum aream sectoris etiam mutationis vestigia in antecedentibus exstant. Inter ipsas enim functiones invenimus æquationes algebraicas, quarum ope ipsarum quæcunque per aliam quamcunque exprimi potest, ex. gr. cofunctiones per suas primarias: ut

$$T^c = \frac{ab}{T},\ \mathfrak{T}^c = \frac{ab}{\mathfrak{T}},\ S^c = a.\sqrt{\frac{c^2 - S^2}{a^2 + S^2}},\ \dot{S}^c = b.\sqrt{\frac{c^2 - \dot{S}^2}{b^2 + \dot{S}^2}},$$

$$\mathfrak{S}^c = a.\sqrt{\frac{\mathfrak{S}^2 - c^2}{\mathfrak{S}^2 - a^2}},\ \dot{\mathfrak{S}}^c = b.\sqrt{\frac{\dot{\mathfrak{S}}^2 + c^2}{\dot{\mathfrak{S}}^2 - b^2}};\ \text{nec non}\ \mathfrak{T} = a.\sqrt{\frac{b^2 + T^2}{a^2 + T^2}},$$

$$S = \frac{cT}{\sqrt{b^2 + T^2}} = a.\sqrt{\frac{c^2 - S^{c2}}{a^2 - S^{c2}}} = \frac{c.\sqrt{\dot{\mathfrak{S}}^2 - b^2}}{\dot{\mathfrak{S}}},\ \&c.$$

Est vero in signis fusius expressis $S^c = S^c\varphi = S^c\left(\frac{a}{b}, \varphi\right) = S\left(\frac{a}{b}, (A - \varphi)\right)$ & $S = S\left(\frac{a}{b}, \varphi\right)$, ideoque juxta alterutram formularum præcedentium

$$S\left(\frac{a}{b}, A - \varphi\right) = a.\sqrt{\frac{c^2 - S\left(\frac{a}{b}, \varphi\right)^2}{a^2 - S\left(\frac{a}{b}, \varphi\right)^2}},\ \text{similiterque}$$

$$\dot{S}\left(\frac{a}{b}, A - \varphi\right) = b.\sqrt{\frac{c^2 - \dot{S}(a, b, \varphi)^2}{b^2 + \dot{S}\left(\frac{a}{b}, \varphi\right)^2}};$$ ex quo videmus $S(\psi - \varphi)$ per $S\varphi$, saltem cum $\psi = A$ fuerit, separatim exprimi posse, exsistente S

sinu alterutro hypercyclico; & simile de secantibus & reliquis concluditur. Sic etiam invenimus $T\overparen{\varphi + \iota B} = \iota\mathfrak{T}\varphi$, ideoque $T\overparen{\varphi + \iota B} =$ $= \iota a . \sqrt{\frac{b^2 + T\varphi^2}{a^2 + T\varphi^2}}$, ex quo simile quid de tangente exspectare licet. Est etiam $T(A - \varphi) = T^c\varphi = \frac{ab}{T\varphi}$. Inde vero universim divinamur, tum tangentem hypercyclicam, tum reliquas functiones (F), æque ac trigonometricas, separatim exprimi posse, ita ut ex. gr. $F(\varphi \pm \psi)$ per $F\varphi$ $F\psi$ algebraice habeatur, præsertim cum jam insignem harum functionum similitudinem invenerimus. Quoniam vero $T(\psi - \varphi) = \frac{ab}{T\varphi}$, quando $\psi = A$, atque $T\overparen{\psi - \varphi} = a\,\iota \sqrt{\frac{b^2 + T\varphi^2}{a^2 + T\varphi^2}} =$

$= a\iota \frac{\sqrt{(b^2 + T\varphi^2)(a^2 + T\varphi^2)}}{a^2 + T\varphi^2}$, cum $\psi = \iota B$ fuerit, jam inde conjicere licet, universim $T(\psi - \varphi)$ sub formam

$\frac{(\psi)_{,} . \sqrt{(a^2 + T\varphi^2)(b^2 + T\varphi^2)} + (\psi)_{,,} . T\varphi}{a + (\psi)_{,,,} . T\varphi^2}$ evolutum iri, exsistentibus $(\psi)_{,}$, $(\psi)_{,,}$ & $(\psi)_{,,,}$ certis ipsius ψ functionibus. Quoniam vero in trigonometricis est $T(\psi - \varphi) = \frac{T\psi - T\varphi}{1 + T\psi . T\varphi}$, seu ducendo utrimque in $1 - T\psi . T\varphi$, ut in denominatore $T\varphi^2$ adsit,

$T\overparen{\psi - \varphi} = \frac{T\psi(1 - T\psi T\varphi) - T\varphi(1 - T\psi T\varphi)}{1 - T\psi^2 T\varphi^2}$, seu

$T\overparen{\psi - \varphi} = \frac{T\psi . (1 + T\varphi^2) - T\varphi . (1 + T\psi^2)}{1 - T\psi^2 T\varphi^2}$; videamus, num aliqua mutatione facta hæc formula ultra trigonometriam extendi possit. Prævie

vero, quia $1 + T\varphi^2 = \mathfrak{d}\, T\varphi = \frac{dT\varphi}{d\varphi}$, hunc valorem sicuti & quantitates constantes arbitrarias $(\alpha,\ \beta)$ introducamus, ut habeatur formula

$T(\psi - \varphi) = \frac{T\psi\, \mathfrak{d}\, T\varphi - T\varphi . \mathfrak{d}\, T\psi}{\alpha + \beta . T\varphi^2 . T\psi^2}$, de qua jam disquiretur, num valeat de alia quadam functione, ac (ut facit, cum $\alpha = 1 = -\beta$) de tangente circulari, & forsan de hypercyclica. Hunc in finem ponamus $\psi = \varphi + rd\varphi$ eritque $T(\psi - \varphi) = T(rd\varphi)$, seu $T(\psi - \varphi) = C_0 + C_{,}\, \overline{rd\varphi} + C_{,,}\, \overline{r\, d\varphi}^{\,2} + ..$, siquidem (ut maxime probabile) Tx sub formam $C_0 + C_{,}\, x + C_{,,}\, x^2 + ..$ explicari potest. Simul erit ex theoremate Tayloreo
$T\psi = T(\varphi + r\, d\varphi) = T\varphi + r\, d\, T\varphi + \frac{1}{2} r^2\, d^2\, T\varphi + .. = T\varphi + r\, d\varphi \mathfrak{d} T\varphi +$; ideoque $\mathfrak{d}\, T\psi = \mathfrak{d}\, T\varphi + r\, \mathfrak{d}\, \varphi\, \mathfrak{d}^2\, T\varphi +$, positis nempe r & d_φ constantibus ideoque $d\psi = d_\varphi = 1$. Formula vero modo divinata subministrat $(\alpha + \beta\, T\varphi^2 . T\psi^2) . T(\psi - \varphi) = T\psi\, \mathfrak{d}\, T\varphi - T\varphi\, \mathfrak{d}\, T\psi$, seu valoribus jam positis introductis,
$(\alpha + \beta\, T\varphi^2 . T\varphi^2 + \beta\, T\varphi^2 . 2r\, T\varphi\, d\varphi \mathfrak{d} T\varphi + ...) . (C_0 + C_{,} rd\varphi + C_{,,} \overline{rd\varphi}^{\,2} + ..) =$
$= \mathfrak{d}\, T\varphi . (T\varphi + r\, d_\varphi . \mathfrak{d}\, T\varphi + ...) - T\varphi . (\mathfrak{d}\, T\varphi + r\, d\varphi\, \mathfrak{d}^2\, T\varphi + ...)$ seu $= r\, d_\varphi . (\mathfrak{d}\, T\varphi^2 - T\varphi\, \mathfrak{d}^2\, T\varphi) + r^2 \varnothing + ...)$. Jam vero ut hæc æquatio pro quocunque ipsius r & $T\varphi$ valore locum habeat, sit necessario $C_0 = o$, ideoque inde elicitur, si jam breviter T loco ipsius $T\varphi$ scribimus, (nempe ex coefficiante ipsius $r . d\varphi$), $C_{,} . (\alpha + \beta\, T^4) = \mathfrak{d} T^2 - T . \mathfrak{d}^2\, T$, cujus jam æquationis differentialis ope functio T determinatur. Ponamus (ad hanc resolvendam) $-C_{,} . R = \frac{d\, T}{T\, d\varphi} = \frac{\mathfrak{d} T}{T}$, eritque $-C_{,}\, \mathfrak{d}\, R = \frac{T \mathfrak{d}^2\, T - \mathfrak{d}\, T^2}{T^2}$

ideoque $\alpha + \beta\, T^4 = T^2\, \mathfrak{d}\, R = T^2 . \frac{dR}{d\varphi}$. Est vero $T\, d\varphi = \frac{-d\, T}{C_{,} R}$,

ideoque $\alpha + \beta\, T^4 = - T^3 \frac{. C_{,} R\, d\, R}{d\, T}$, seu $-2 . C_{,} R\, d\, R =$